

# RENORMALIZATION AND FORCING OF HORSESHOE ORBITS

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**ABSTRACT.** In this paper we deal with the Boyland order of horseshoe orbits. We prove that there exists a set  $\mathcal{R}$  of renormalizable horseshoe orbits containing only quasi-one-dimensional orbits, that is, for these orbits the Boyland order coincides with the unimodal order.

## 1. INTRODUCTION

In [2], Boyland introduced the forcing relation between periodic orbits of the disk  $D^2$ . Given two periodic orbits  $P$  and  $R$ , we say that  $P$  *forces*  $R$ , denoted by  $P \geq_2 R$ , if every homeomorphism of  $D^2$  containing the braid type of  $P$  must contain the braid type of  $R$ . The set of periodic orbits forced by  $P$  is denoted by  $\Sigma_P$ . In this paper we are concerned with the forcing of Smale horseshoe periodic orbits. A horseshoe orbit  $P$  is called *quasi-one-dimensional* if  $P$  forces all orbit  $R$  such that  $P \geq_1 R$ , where  $\geq_1$  is the unimodal order. In [4], Hall gave a set of quasi-one-dimensional horseshoe orbits, called NBT orbits (Non-Bogus Transition orbits) which are in bijection with  $\mathbb{Q} \cap (0, \frac{1}{2})$  and have the property that their thick interval map induced has minimal periodic orbit structure, that is, if  $P$  is an NBT orbit then every braid type of a periodic orbit of its thick interval map  $\theta_P$  is forced by the braid type of  $P$ .

In this paper we obtain a type of orbits which are quasi-one-dimensional too although their associated thick interval maps are reducible in the sense of Thurston [6], that is, they are isotopic to reducible homeomorphisms which have an invariant set of non-homotopically trivial disjoint curves  $\{C_1, \dots, C_n\}$ . Restricted to the components of  $D^2 \setminus \{C_1, \dots, C_n\}$ , these reducible maps (or one of its power) have minimal periodic orbit structure.

**Theorem 1.** *There exists a set  $\mathcal{R} \supset \text{NBT}$  of quasi-one-dimensional horseshoe orbits, that is, if  $P \in \mathcal{R}$  then  $\Sigma_P = \{R : P \geq_1 R\}$ .*

These orbits are defined using the renormalization operator which was introduced in [3] as the  $*$ -product.

## 2. PRELIMINARIES

**2.1. Boyland partial Order.** Let  $D_n$  be the punctured disk. Let  $\text{MCG}(D_n)$  be the group of isotopy classes of homeomorphisms of  $D_n$ , which is called the *mapping class group* of  $D_n$ . Given a homeomorphism  $f : D^2 \rightarrow D^2$  of the disk  $D^2$  with a periodic orbit  $P$ , the *braid type* of  $P$ , denoted by  $\text{bt}(P, f)$ , is defined as follows: Take an orientation preserving homeomorphism  $h : D^2 \setminus P \rightarrow D_n$  then  $\text{bt}(P, f)$  is the conjugacy class  $[h \circ f \circ h^{-1}] \in \text{MCG}(D_n)$  of  $h \circ f \circ h^{-1} : D_n \rightarrow D_n$ .

Let  $\text{BT}$  be the union of all the periodic braid types and let  $\text{bt}(f)$  be the set formed by the braid types of the periodic orbits of  $f$ . We will say that  $f : D^2 \rightarrow D^2$  *exhibits* a braid type  $\beta$  if there exists an  $n$ -periodic orbit  $P$  for  $f$  with  $\beta = \text{bt}(P, f)$ . Now we can define the relation  $\geq_2$  on  $\text{BT}$ . We say that  $\beta_1$  *forces*  $\beta_2$ , denoted by  $\beta_1 \geq_2 \beta_2$ , if every homeomorphism exhibiting  $\beta_1$ , exhibits  $\beta_2$  too. Then it is said that a periodic orbit  $P$  *forces* another periodic orbit  $R$ , denoted by  $P \geq_2 R$ , if  $\text{bt}(P) \geq_2 \text{bt}(R)$ .

In [2], P. Boyland proved the following theorem.

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**Theorem 2.** [2, Theorem 9.1] *The relation  $\geq_2$  is a partial order.*

**2.2. Smale horseshoe.** The Smale horseshoe is a map  $F : D^2 \rightarrow D^2$  of the disk which acts as in Fig. 1. The set  $\Omega = \bigcap_{j \in \mathbb{Z}} F^j(V_0 \cup V_1)$  is  $F$ -invariant and  $F|_\Omega$  is conjugated to the shift  $\sigma$  on the sequence space of two symbols 0 and 1,  $\Sigma_2 = \{0, 1\}^{\mathbb{Z}}$ , where

$$(1) \quad \sigma((s_i)_{i \in \mathbb{Z}}) = (s_{i+1})_{i \in \mathbb{Z}}.$$

The conjugacy  $h : \Omega \rightarrow \Sigma_2$  is defined by

$$(2) \quad (h(x))_i = \begin{cases} 0 & \text{if } F^i(x) \in V_0, \\ 1 & \text{if } F^i(x) \in V_1. \end{cases}$$

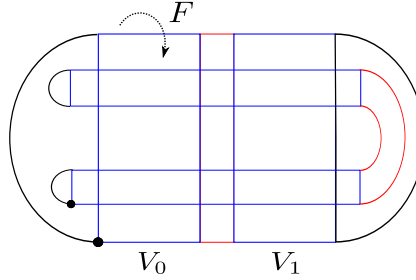


FIGURE 1. Dynamics of  $F$ .

To compare horseshoe orbits it is necessary to define the *unimodal order*. It is a total order in  $\Sigma^+ = \{0, 1\}^{\mathbb{N}}$  given by the following rule: Let  $s = s_0 s_1 \dots$  and  $t = t_0 t_1 \dots$  be sequences in  $\Sigma^+$  such that  $s_i = t_i$  for  $i \leq k$  and  $s_{k+1} \neq t_{k+1}$ , then  $s < t$  if

- (O1)  $\sum_{i=0}^k s_i$  is even and  $s_{k+1} < t_{k+1}$ , or
- (O2)  $\sum_{i=0}^k s_i$  is odd and  $s_{k+1} > t_{k+1}$ .

We say that  $s \geq_1 t$  if either  $s = t$  or  $s > t$ .

Every  $n$ -periodic orbit  $P \in \Omega$  of  $F$  has a *code* denoted by  $c_P \in \Sigma_2$ . It is obtained from  $h(p) = c_P^\infty$  where  $p$  is a point of  $P$  and  $c_P$  satisfies  $\sigma^n(c_P) \leq_1 c_P$ , that is,  $c_P$  is *maximal* in the unimodal order  $\geq_1$ . We say that  $P \geq_1 R$  if  $\sigma^n(R) \leq_1 c_P$ ,  $\forall n \geq 1$ . For every orbit  $P$ , there exists a homeomorphism  $\theta_P$  that realizes the combinatorics of  $P$ . This is obtained fattening the line diagram of  $P$  and it is called the *tick map induced by  $P$* . See [4].

**2.3. Renormalized Horseshoe Orbits.** Let  $P$  and  $Q$  be two horseshoe periodic orbits with codes  $c_P = A a_{n-1}$  where  $A = a_0 a_1 \dots a_{n-2}$  and  $c_Q = b_0 b_1 \dots b_{m-2} b_{m-1}$  with periods  $n$  and  $m$ , respectively.

**Definition 3** (Renormalization Operator). We will write  $P * Q$  for the  $nm$ -periodic orbit with code

$$(3) \quad c_{P*Q} = \begin{cases} A b_0 A b_1 A b_2 \dots A b_{m-2} A b_{m-1} & \text{if } \epsilon(A) \text{ is even} \\ A \bar{b}_0 A \bar{b}_1 A \bar{b}_2 \dots A \bar{b}_{m-2} A \bar{b}_{m-1} & \text{if } \epsilon(A) \text{ is odd} \end{cases}$$

where  $\epsilon(A) = \sum_{i=0}^{n-2} a_i$  and  $\bar{b}_i = 1 - b_i$ .

The orbit  $P * Q$  is called the *renormalization of  $P$  and  $Q$* . If an orbit  $S$  satisfies  $S = P * Q$  for some  $P, Q \in \Sigma_2$ , it is said that  $S$  is *renormalizable*. Also we will denote

$$P_1 * P_2 * \dots * P_k = (\dots ((P_1 * P_2) * P_3) \dots).$$

**Example 4.** If  $P = 101$  and  $Q = 1001$  then  $P * Q = 100101101100$ .

**2.4. NBT Orbits.** There are a type of horseshoe orbits for which the Boyland partial order is well-understood. They are constructed in the following way. Given a rational number  $q = \frac{m}{n} \in \widehat{\mathbb{Q}} := \mathbb{Q} \cap (0, \frac{1}{2})$ , let  $L_q$  be the straight line segment joining  $(0, 0)$  and  $(n, m)$  in  $\mathbb{R}^2$ . Then construct a finite word  $c_q = s_0 s_1 \cdots s_n$  as follows:

$$(4) \quad s_i = \begin{cases} 1 & \text{if } L_q \text{ intersects some line } y = k, k \in \mathbb{Z}, \text{ for } x \in (i-1, i+1) \\ 0 & \text{otherwise} \end{cases}$$

It follows that  $c_q$  is palindromic and has the form:

$$(5) \quad c_q = 10^{\mu_1} 1^2 0^{\mu_2} 1^2 \cdots 1^2 0^{\mu_{m-1}} 1^2 0^{\mu_m} 1.$$

We will denote  $P_q$  to the periodic orbits of period  $n+2$  which have the codes  $c_{q0}^1$ , when the distinction is not important and let  $\text{NBT} = \{P_q : q \in \widehat{\mathbb{Q}}\}$ . In [4], Hall proved the following result.

**Theorem 5.** *Let  $q, q' \in \widehat{\mathbb{Q}}$ . Then*

- (i)  $P_q$  is quasi-one-dimensional, that is,  $P_q \geq_1 R \implies P_q \geq_2 R$ .
- (ii)  $q \leq q' \iff (c_{q1}^0)^\infty \geq_1 (c_{q'1}^0)^\infty \iff (c_{q1}^0)^\infty \geq_2 (c_{q'1}^0)^\infty$

So theorem above says that the Boyland order restricted to the NBT orbits is equal to the unimodal order.

### 3. FORCING OF RENORMALIZABLE ORBITS

For proving Theorem 1 we need the following result:

**Theorem 6.** *Let  $P = A_1^0$  and  $Q$  be periodic orbits. Then*

$$(6) \quad \Sigma_{P*Q} = \{R : P \geq_2 R\} \cup \{P * R : Q \geq_2 R\}.$$

To prove the result above it will be needed two lemmas whose proofs are left to the reader.

**Lemma 7.** *Let  $i, j \in \{1, \dots, n-1\}$  be positive integers with  $i \neq j$  and  $T_M = P * Q$  and  $T_m = \sigma^n(T_M)$ . Then*

- (a) if  $\epsilon(A)$  is even then  $A0^\infty \leq_1 T_m \leq_1 T_M \leq_1 A1^\infty$ ,
- (b) if  $\epsilon(A)$  is odd then  $A1^\infty \leq_1 T_m \leq_1 T_M \leq_1 A0^\infty$ ,
- (c)  $\sigma^i(P) \leq_1 \sigma^j(P) \iff [\sigma^i(T_M) \leq_1 \sigma^j(T_M) \text{ and } \sigma^i(T_m) \leq_1 \sigma^j(T_m)]$ .

**Lemma 8.** *Let  $P = A_1^0$  and  $Q$  be two periodic orbits. If  $i, j \in \{0, \dots, m-1\}$ , with  $i \neq j$ , then*

$$\sigma^i(Q) >_1 \sigma^j(Q) \iff \sigma^{in}(P * Q) >_1 \sigma^{jn}(P * Q).$$

*Proof of Theorem 6.* Let  $\theta_{P*Q}$  be the thick map induced by  $P * Q$ . First we see that the only iterates of  $P * Q$  satisfying  $T_m \leq_1 \sigma^i(P * Q) \leq_1 T_M$  are the iterates  $\sigma^{in}(P * Q)$ , with  $0 \leq i \leq m-1$ ; so there exists a curve  $C_{n-1}$  containing these orbits disjoint from the others and bounding a region  $D_{n-1}$ . By Lemma 7(c) and noting that  $\sigma^i(T_m)$  and  $\sigma^i(T_M)$  has the same initial symbol for  $i \in \{1, \dots, n-1\}$ , it follows that  $\{\theta_{P*Q}^i(C_{n-1})\}_{i=1}^{n-1}$  has the same combinatorics as  $P$ . For  $i = 0, \dots, n-2$ , let  $C_i = \theta_{P*Q}^{i+1}(C_{n-1})$  be a curve which bounds a domain  $D_i$ . It is possible to define  $\theta_{P*Q}$  such that  $\theta_{P*Q}^n(C_{n-1}) = C_{n-1}$ . Then the line diagram of  $\{D_0, \dots, D_{n-1}\}$  is as the line diagram of  $P$  and then  $\theta_{P*Q}$  has the same behaviour than  $\theta_P$  in the exterior of  $\cup D_i$ . Since  $\theta_{P*Q}$  can be reduced by a family of curves, we will need study the Thurston representative of  $\theta_{P*Q}$  restricted to  $D^2 \setminus \cup C_i$ . As  $\theta_P$  and  $\theta_{P*Q}$  have the same combinatorics in the exterior of  $\cup D_i$ , they have the same Thurston representative in the exterior of  $\cup D_i$ . So  $P$  and  $P * Q$  force the same periodic orbits in the exterior of  $\cup D_i$ . Then  $\{R : P \geq_2 R\} \subset \Sigma_{P*Q}$ . See Fig. 2.

It is clear that to find what orbits are forced by  $P * Q$  in  $\cup D_i$ , it is enough to study  $\theta_{P*Q}^n$  restricted to  $D_{n-1}$ . By Lemma 8, the line diagram of  $\theta_{P*Q}^n$  inside  $D_{n-1}$  is the same as the line

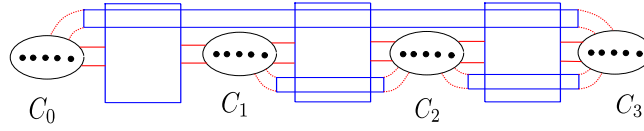
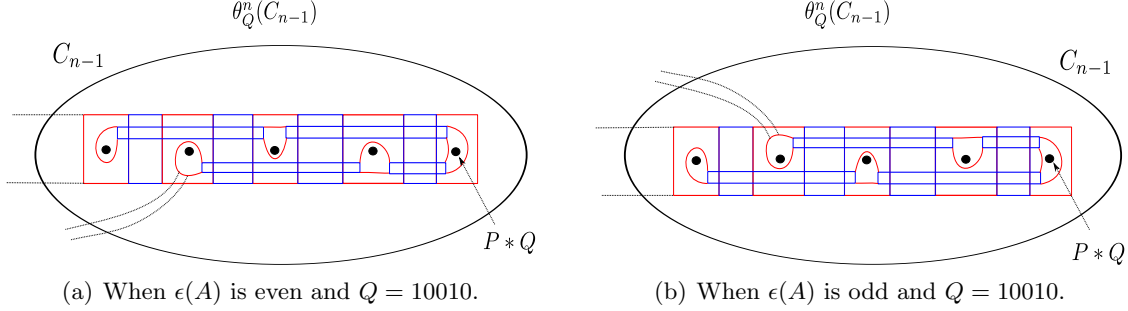
FIGURE 2. The image  $\theta_{P*Q}(C_i)$  when  $P = 1001$ .FIGURE 3. The image  $\theta_{P*Q}^n(C_{n-1})$ .

diagram of  $Q$  when  $\epsilon(A)$  is even, and it is flipped when  $\epsilon(A)$  is odd. See Fig. 3. So  $\theta_{P*Q}^n$  has the same combinatorics than  $\theta_Q$ .

As in Lemma 8, we can prove that  $R \leq_2 Q \iff P * R \leq_2 P * Q$ . So  $\Sigma_{P*Q} = \{R : P \geq_2 R\} \cup \{P * R : Q \geq_2 R\}$ .  $\square$

*Remark 9.* Theorem 6 says us that to look for the orbits that are forced by  $P * Q$  it is enough to look for the orbits that are forced by  $P$  and the orbits that are forced by  $Q$ . So we can study the thick maps induced by  $P$  and  $Q$  separately. Every of these thick maps can be reduced using methods to determine its minimal representative, e.g. [1, 5].

**Corollary 10.** *Let  $P_1, P_2, \dots, P_k$  be NBT orbits. Then*

- (a)  $\Sigma_{P_1 * \dots * P_k} = \bigcup_{j=1}^k \{P_1 * \dots * P_{j-1} * R : P_j \geq_1 R\}$
- (b)  $\Sigma_{P_1 * \dots * P_k} = \{R : P_1 * \dots * P_k \geq_1 R\}$

*Proof.* Item (a) follows directly from Theorem 6. For item (b) it is enough to prove that if  $P$  and  $Q$  are quasi-one-dimensional horseshoe orbits then  $P * Q$  is a quasi-one-dimensional orbit too. Suppose that  $\epsilon(A)$  is even. From Theorem 6,

$$(7) \quad \Sigma_{P*Q} = \{R : P \geq_1 R\} \cup \{P * R : Q \geq_1 R\}.$$

Hence it follows that  $\Sigma_{P*Q} \subset \{S : P * Q \geq_1 S\}$ . We have to prove the inclusion  $\{S : P * Q \geq_1 S\} \subset \Sigma_{P*Q}$ . If  $P \geq_1 S$  then  $S \in \Sigma_{P*Q}$ . Let  $S$  with  $c_S = s_0 s_1 \dots s_{k-1}$  be a periodic orbit with  $P \leq_1 S \leq_1 P * Q$ . By Lemma 7(a),  $(A0)^\infty \leq_1 c_S^\infty \leq_1 (A1)^\infty$ . This implies that  $c_S = A s_{n-1} s_n \dots$  and

$$(0A)^\infty \leq_1 \sigma^{n-1}(c_S^\infty) = s_{n-1} s_n \dots \leq_1 (1A)^\infty,$$

and  $\sigma^n(c_S^\infty) \geq_1 (A0)^\infty$ . In the other hand  $\sigma^n(c_S^\infty) \leq_1 c_{P*Q}^\infty$ . Then  $\sigma^n(c_S^\infty) = A s_{2n-1} \dots$  and then  $c_S = A s_n A s_{2n-1} \dots$ . Continuing this process, it follows that  $S = P * R$  where  $c_R = s_{n-1} s_{2n-1} \dots$ . So  $P * R \leq_1 P * Q$  which implies that  $R \leq_1 Q$ . So  $S \in \{P * R : Q \geq_1 R\}$  and the proof is finished.  $\square$

Now we proceed to prove Theorem 1.

*Proof of Theorem 1.* Let  $\{P_j\}_{j \in \mathbb{N}}$  be the set of NBT orbits and consider the space  $\mathbb{N}^{\mathbb{N}}$  of sequences of positive integers. Take a sequence  $\mathcal{J} = (j_1, j_2, \dots, j_n, \dots) \in \mathbb{N}^{\mathbb{N}}$  and define

$$(8) \quad \mathcal{R}_{\mathcal{J}} = \cup_{k=1}^{\infty} \{P_{j_1} * \dots * P_{j_k}\}$$

and  $\mathcal{R} = \cup_{\mathcal{J} \in \mathbb{N}^{\mathbb{N}}} \mathcal{R}_{\mathcal{J}}$ . By Corollary 10(b), every orbit of  $\mathcal{R}$  is quasi-one-dimensional.  $\square$

**Example 11.** By previous Theorem, if  $S = 10010 * 10010 = 1001110010100101001110010$ ,

$$(9) \quad \Sigma_S = \{R : 10010 \geq_1 R\} \cup \{10010 * R : 10010 \geq_1 R\}.$$

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